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LETTER TO THE EDITOR

Stability of the Minkowski vacuum in the renormalised semiclassical theory of gravity

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Abstract. By making use of the dimensional regularisation and renormalisation group arguments we show that the conclusion of Horowitz with regard to the instability of flat space–time in the semiclassical theory of gravity also holds for the coupling of a massive real scalar quantum field to the classical gravity $g_{\mu\nu}$, but only on condition that the mass of the field is not too large.

The semiclassical theory of gravity is defined to be a theory in which all physical fields are quantised except for the gravity whose dynamics is governed by the Einstein field equation

$$G_{\mu\nu}(x) = -8\pi g_{\rm B}^{-1}\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle. \tag{1}$$

Here $g_{\rm B}^{-1}$ is the 'bare' Newtonian coupling constant, $G_{\mu\nu}(x) = R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)R(x)$ is the Einstein tensor constructed from the classical metric tensor $g_{\mu\nu}$ and $\langle\psi|\hat{T}_{\mu\nu}(x)|\psi\rangle$ is the expectation value of the stress tensor of the quantised fields in the normalised state $|\psi\rangle$.

In a recent paper Horowitz (1980, 1981) has shown that if one couples a *massless* quantum field to classical gravity via equations (1) and assumes that the observed cosmological constant is zero then the infinitesimal perturbations of the Minkowski ground state $(\eta_{\mu\nu};\, |0, in\rangle)$ will not be oscillatory about this constant solution but will grow exponentially in time. His result is based on an indirect evaluation of the regularised $\langle 0, in|\hat{T}_{\mu\nu}(x)|0, in\rangle$ from a set of five plausible axioms.

In this letter we apply the scheme of dimensional regularisation to the coupling of a *massive* real scalar field $\hat{\phi}$ to the classical gravity. Then, in addition to recovering the results of Horowitz in the massless limit, we prove that by imposing a positivity condition on the renormalised coupling of the $R_{\mu\nu}R^{\mu\nu}$ term one may retain stability but only for masses well above the Planck mass! The addition of a $\sqrt{-g}R\phi^2$ term to the Lagrangian puts strict lower and upper bounds on the scalar field mass beyond which the theory becomes unstable. However these bounds—being of the order of Planck mass—are too high to be acceptable for realistic physical fields.

To start let us assume that the dynamics of $\hat{\phi}$ is governed by

$$\left(\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu) - \mu^2\right)\hat{\phi}(x) = 0. \tag{2}$$

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The Heisenberg picture $\hat{T}_{\mu\nu}(x)$ is formally given by

$$\hat{T}_{\mu\nu}(x) = -\partial_\mu\hat{\phi}(x)\partial_\nu\hat{\phi}(x) + \tfrac{1}{2}g_{\mu\nu}(x)(g^{\lambda\sigma}(x)\partial_\lambda\hat{\phi}(x)\partial_\sigma\hat{\phi}(x) + \mu^2\hat{\phi}^2(x)). \tag{3}$$

In a previous paper (Kibble and Randjbar–Daemi 1980) we have given a variational principle which yields the Schrödinger picture version of the system (1) and (2). This action integral may be generalised to include the higher-order terms $\sqrt{-g}R^2$ and $\sqrt{-g}R_{\mu\nu}R^{\mu\nu}$ which will be induced by the requirement of renormalisation. Let us consider the action integral

$$W(g_{\mu\nu}, |\psi\rangle) = \frac{1}{(4\pi)^{n/2}}\int \mathrm{d}^n x\sqrt{-g}\{\Lambda_\mathrm{B} + g_\mathrm{B}R + \lambda_\mathrm{B}R^2 + \gamma_\mathrm{B}R_{\mu\nu}R^{\mu\nu}\} + W_\psi \tag{4}$$

where B stands for 'bare' and in anticipation of the dimensional regularisation scheme which we are going to employ we carry out all of our calculations in an n-dimensional Riemannian manifold. It is only after the renormalisation of the theory that we continue back to the physical four-dimensional pseudo-Riemannian manifold with the signature $(-, +, +, +)$. W_ψ stands for the action of the matter fields (Kibble and Randjbar–Daemi 1980).

One can develop a perturbative scheme to evaluate the diagonal matrix elements of (3) in any normalisable state $|\psi\rangle$. This and the other details of the calculation will be presented elsewhere. Here we only give the results of a relatively long calculation for the choice $|0, in\rangle$ of the quantum state $|\psi\rangle$ and the linearised $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We shall neglect any term higher than the first order in $h_{\mu\nu}$. Also, since we are investigating the stability of the solutions discussed by Horowitz (1980, 1981) we shall set $h = \eta^{\mu\nu}h_{\mu\nu} = 0$. Then in the harmonic gauge $\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0$ the scalar curvature $R = 0$ and the Ricci tensor is given by $R_{\mu\nu} = -\frac{1}{2}\partial^2 h_{\mu\nu}$. Upon insertion of $h(x) = 0$ in the regularised $\langle 0, in|\hat{T}_{\mu\nu}(x)|0, in\rangle$ we obtain

$$\langle 0, in|\hat{T}_{\mu\nu}(x)|0, in\rangle^{\mathrm{reg}} = \frac{1}{\varepsilon}\frac{m^{-2\varepsilon}}{(4\pi)^{n/2}} \times\left(\tfrac{1}{4}(1+\tfrac{3}{2}\varepsilon)\frac{\mu^4}{\mu_\mathrm{R}^{2\varepsilon}}g_{\mu\nu}(x) + \frac{1}{120}\partial^4 h_{\mu\nu}(x) - \frac{\mu^2}{12}\partial^2 h_{\mu\nu}(x) + \varepsilon\phi_{\mu\nu}(x)\right) \tag{5}$$

where $\varepsilon = 2 - \frac{1}{2}n$ and the function $\phi_{\mu\nu}(x)$ becomes finite as $\varepsilon \to 0$. Here m is an arbitrary unit of mass such that $\mu_\mathrm{B} = \mu_\mathrm{R} m$. Since one does not need a ϕ^2 counter term, the bare mass μ_B is already finite. Thus we drop the suffix B from μ_B. The dimensionless μ_R has been introduced only for later convenience. However, in order to discuss the scaling properties of the renormalised theory under variations of the scale of mass it is essential to introduce the unit of mass m ('t Hooft 1973).

Let us insert the above $\langle 0, in|\hat{T}_{\mu\nu}(x)|0, in\rangle$ into the linearised Einstein equation $\delta W/\delta g^{\mu\nu} = 0$, with W given as in (4). We get

$$\tfrac{1}{2}[\Lambda_\mathrm{B} + \tfrac{1}{4}(1+\tfrac{3}{2}\varepsilon)(\mu^{4-2\varepsilon}/\varepsilon)]g_{\mu\nu}(x) + \left(g_\mathrm{B} + \frac{\mu_\mathrm{R}^2 m^{2-2\varepsilon}}{12\varepsilon}\right)G_{\mu\nu}(x) - \frac{1}{2}\left(\gamma_\mathrm{B} - \frac{m^{-2\varepsilon}}{120\varepsilon}\right)\partial^4 h_{\mu\nu}(x) + \tfrac{1}{2}\phi_{\mu\nu}(x) = 0. \tag{6}$$

We have argued elsewhere (Randjbar-Daemi *et al* 1980) that in a semiclassical field theory one may employ the dynamics of the classical field to remove the finite renormalisation ambiguity of the physical parameters. For instance if one demands—as we shall do—that an initial state comprising $g_{\mu\nu} = \eta_{\mu\nu}$ and $|\psi\rangle = |0, in\rangle$ will remain the same, then one must have

$$\Lambda_{\rm B} + \tfrac{1}{4}(1 + \tfrac{3}{2}\varepsilon)(\mu^{4-2\varepsilon}/\varepsilon) = 0. \tag{7}$$

Let us also introduce the dimensionless renormalised parameters $g_{\rm R}$ and $\gamma_{\rm R}$ through the following definitions

$$g_{\rm B} = m^{2-2\varepsilon}\left(g_{\rm R} - \frac{\mu_{\rm R}^2}{12\varepsilon}\right) \tag{8a}$$

$$\gamma_{\rm B} = m^{-2\varepsilon}\left(\gamma_{\rm R} + \frac{1}{120\varepsilon}\right). \tag{8b}$$

Of course these definitions are an expression of the minimal subtraction scheme. The fact that the bare parameters $\mu_{\rm B}$, $g_{\rm B}$ and $\gamma_{\rm B}$ are independent of the unit of mass m forces $\mu_{\rm R}$, $g_{\rm R}$ and $\gamma_{\rm R}$ to depend on m, such that if one introduces a scaling parameter s through $m = m_0 \exp(s)$ then the renormalised parameters must satisfy the following 'β-function' equations

$$\frac{\partial \mu_{\rm R}}{\partial s} = -\mu_{\rm R} \tag{9a}$$

$$\frac{\partial g_{\rm R}}{\partial s} = -2g_{\rm R} - \tfrac{1}{6}\mu_{\rm R}^2 \tag{9b}$$

$$\frac{\partial \gamma_{\rm R}}{\partial s} = \frac{+1}{60}. \tag{9c}$$

These equations guarantee that the change in the scale of m does not alter the prescription of the minimal subtraction. We shall shortly prove that the same equations ensure the scale independence of the renormalised linearised Einstein equations.

One can of course easily integrate the system (9) to obtain

$$\mu_{\rm R}(s) = \mu_{\rm R}(0)\, {\rm e}^{-s} \tag{10a}$$

$$g_{\rm R}(s) = g_{\rm R}(0)\, {\rm e}^{-2s} + \tfrac{1}{6}\mu_{\rm R}^2(s) \lg(\mu_{\rm R}(s)/\mu_{\rm R}(0)) \tag{10b}$$

$$\gamma_{\rm R}(s) = +\tfrac{1}{60}s + \gamma_{\rm R}(0). \tag{10c}$$

After substitution from equations (8) into equation (6) we continue back to the four-dimensional physical space to obtain (assuming that not all of $h_{\mu\nu} = 0$) the following Fourier space equation

$$Aq^4 + Bq^2 + C = \tfrac{16}{15}(\mu^2 + \tfrac{1}{4}q^2)^2(\tfrac{1}{4} + \mu^2/q^2)^{1/2} \lg \frac{(\mu^2 + \frac{1}{4}q^2)^{1/2} + (\frac{1}{4}q^2)^{1/2}}{(\mu^2 + \frac{1}{4}q^2)^{1/2} - (\frac{1}{4}q^2)^{1/2}} \tag{11}$$

where since we are interested in the stability question we have already assumed that $q^2 \geqslant 0$. The coefficients A, B and C are given by

$$A = -4\gamma_{\rm R} - \tfrac{1}{15}\lg \mu_{\rm R} + 46/450 \tag{12a}$$

$$B = 4m^2 g_{\rm R} - \tfrac{2}{3}\mu^2 \lg \mu_{\rm R} + \tfrac{43}{45}\mu^2 \tag{12b}$$

$$C = \tfrac{16}{15}\mu^4. \tag{12c}$$

The right-hand side of equation (11) is manifestly independent of m, whereas the left-hand side depends on m through A and B. However, if we insert from equations (10) into (12) we obtain

$$A = -\tfrac{1}{15}\lg \mu_{\mathrm{R}}(0) - 4\gamma_{\mathrm{R}}(0) + 46/450 \tag{13a}$$

$$B = 4m_0^2 g_{\mathrm{R}}(0) - \tfrac{2}{3}\mu_{\mathrm{R}}^2(0)m_0^2 \lg \mu_{\mathrm{R}}(0) + \tfrac{43}{45}\mu_{\mathrm{R}}^2(0)m_0^2 \tag{13b}$$

$$C = \tfrac{16}{15}m_0^4\mu_{\mathrm{R}}^4(0). \tag{13c}$$

The number $g_{\mathrm{R}}(0)m_0^2$ must of course be related to the Newtonian constant C_{N}, e.g. $g_{\mathrm{R}}(0)m_0^2 = C_{\mathrm{N}}^{-1}$. The parameter $\gamma_{\mathrm{R}}(0)$ on the other hand is absolutely arbitrary. In what follows we shall show that if $0 \leqslant \mu_{\mathrm{R}}(0) < \mathrm{e}^2$ then no finite value of $\gamma_{\mathrm{R}}(0)$ can eliminate the solutions $q^2 \geqslant 0$ of equation (11). However if $\mu_{\mathrm{R}}(0) \geqslant \mathrm{e}^2$ then for $\gamma_{\mathrm{R}}(0) > 0$ the solution $q^2 > 0$ will be absent (here $\mathrm{e} = 2.718\ldots$).

Let us first investigate the case of $\mu_{\mathrm{R}}(0) = 0$. In this limit $B = 4m_0^2 g_{\mathrm{R}}(0)$ and $C = 0$. The coefficient A on the other hand involves the term $-\frac{1}{15}\lg \mu_{\mathrm{R}}(0)$ which will give rise to $-\frac{1}{15}q^2 \lg \mu_{\mathrm{R}}(0)$ on the LHS of equation (11). A close investigation of the RHS of equation (11) reveals that a similar factor is also contained in the RHS of this equation. Thus as $\mu_{\mathrm{R}}(0) \to 0$ equation (11) reduces to

$$ax + b = x \lg x \tag{14}$$

where

$$x = m_0^{-2} q^2 \tag{15a}$$

$$a = 30(-4\gamma_{\mathrm{R}}(0) + 46/450) \tag{15b}$$

$$b = 120 g_{\mathrm{R}}(0) m_0^2. \tag{15c}$$

For $a > 0$ one may make the solution x go to infinity by choosing $\gamma_{\mathrm{R}}(0)$ to be very large and negative. This will correspond to a large value of q^2 and the instability will become a short-range phenomenon. However, for those negative a for which $\gamma_{\mathrm{R}}(0)$ is large and positive the solution x will be small and therefore the instability will become a long-distance phenomenon. This confirms the results obtained by Horowitz for the massless case.

Now let us consider the case of $\mu_{\mathrm{R}}(0) \neq 0$. The RHS of equation (11) is always positive and an increasing function of q^2. Its value at $q^2 = 0$ is $\frac{16}{15}\mu^4$ and its derivative at this point is $\frac{28}{45}\mu^2$. (To check these statements it is convenient to introduce a new variable x through the definition $(q^2)^{1/2} = 2\mu \sinh x$. Then the RHS of (11) can be written as $\frac{16}{15}\mu^4 x \cosh^5 x/\sinh x$.) The value of the LHS of (11) at $q^2 = 0$ is also $\frac{16}{15}\mu^4$. However, its slope at this point depends on the sign of B and its subsequent behaviour depends on the sign of A. The sign of B depends on the value of $\mu_{\mathrm{R}}(0)$ and as is evident from equation (13b) all values of $\mu_{\mathrm{R}}(0) \leqslant 1$ will make B positive. Thus for all masses below the Planck mass m_0 there will always be solutions for $q^2 > 0$ whatever the sign of A. If we set $4g_{\mathrm{R}}(0) = 1$ then $\mu_{\mathrm{R}}(0)$ must roughly be e^2 in order for B to be negative. In this case equation (13a) indicates that any $\gamma_{\mathrm{R}}(0) \geqslant 0$ will make A negative. Therefore one obtains the qualitative behaviour of either side of equation (11) as shown in figure 1. The inclusion of $(4\pi)^{-n/2}\int \mathrm{d}^n x\sqrt{-g}\xi R\phi^2$ with $\xi = (2-n)/4(n-1)$ in the action integral (4) leaves A and C the same as in equation (13) but modifies the values of B as follows:

$$B^{\text{modified}} = 4m_0^2 g_{\mathrm{R}}(0) + 10\mu_{\mathrm{R}}^2(0)m_0^2 \lg \mu_{\mathrm{R}}(0) - \tfrac{13}{5}\mu_{\mathrm{R}}^2(0)m_0^2.$$

Again if we set $4g_{\mathrm{R}}(0) = 1$ then one can easily check that B is negative if $\mu_{\mathrm{R}}(0)$ lies in the range $\mathrm{e}^{-3/2} < \mu_{\mathrm{R}}(0) < \mathrm{e}^{1/4}$. By choosing the coefficient of $\sqrt{-g}R\phi^2$ different from

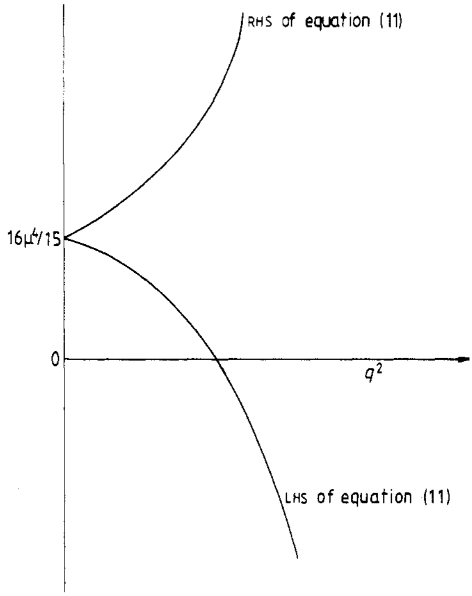

Figure 1. The qualitative behaviour of equation (11) for $\mu_R(0) \geqslant e^2$ and $\gamma_R(0) \geqslant 0$, i.e. the range of no solution for equation (11).

$(2-n)/4(n-1)$ one might be able to improve on this result. But then one loses other nice features.

In addition to this undesirable feature of the semiclassical theory of gravity mention must also be made of the nonlinearity of quantum mechanics (Kibble 1981). This and the details of the calculations of the present paper will be discussed in a forthcoming paper (Randjbar–Daemi 1981).

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References

't Hooft G 1973 *Nucl. Phys.* B **61** 455
Horowitz G T 1980 *Phys. Rev.* D **21** 1450
—— 1981 *Proc. 1980 Oxford Quantum Gravity Conference* to be published
Kibble T W B 1981 *Proc. 1980 Oxford Quantum Gravity Conference* to be published
Kibble T W B and Randjbar-Daemi S 1980 *J. Phys. A: Math. Gen.* **13** 141
Randjbar-Daemi S 1981 to be published
Randjbar-Daemi S, Kay B S and Kibble T W B 1980 *Phys. Lett.* **91B** 417